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Intertwining technique for a system of difference Schrödinger equations and new exactly solvable multichannel potentials

L M Nieto[1], **B F Samsonov**[1,2] **and A A Suzko**[3]

[1] Departamento de Física Teórica, Universidad de Valladolid, 47005 Valladolid, Spain
[2] Department of Quantum Field Theory, Tomsk State University, 36 Lenin Ave, 634050 Tomsk, Russia
[3] Joint Institute of Nuclear Technology and Energetics, National Academy of Sciences of Belarus, Minsk, Belarus

E-mail: luismi@metodos.fam.cie.uva.es, samsonov@phys.tsu.ru and suzko@jinr.ru

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Abstract
The intertwining operator technique is applied to difference Schrödinger equations with operator-valued coefficients. It is shown that these equations appear naturally when a discrete basis is used for solving a multichannel Schrödinger equation. New families of exactly solvable multichannel Hamiltonians are found.

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1. Introduction

The method for finding approximate solutions to the Schrödinger equation based on a matrix representation for the Hamiltonian is widely used in nonrelativistic quantum mechanics [1]. Its rationale is the intuitive idea that the larger the finite basis used in particular calculations, the more exact are the results that may be achieved, and the 'most exact' solution of the problem corresponds to the entire (infinite) basis set. Of course, there is now a rigorous justification for this physically intuitive idea within functional analysis [2], but we would like to stress here that up to now only a few cases are known where one is able to manipulate a whole Fourier series expansion and get the exact solution to the problem. In particular, the well-known J-matrix method [3] (although it has recently been significantly modified [4]), uses a special truncation procedure for modelling a real scattering process involving interacting particles. Until now this method has used mainly the possibility of getting exact solutions to the simplest quantum mechanical problem, namely solving the free particle Schrödinger equation expressed in a special basis as the eigenvalue problem for a tridiagonal (Jacobi) matrix.

Recently [5] it has been shown that an interaction can be included in such a model without affecting the tridiagonal form of the Hamiltonian, a fact that can significantly enlarge the range of possible applications for the J-matrix method. In this paper, we generalize the results of [5] to the case of a matrix Schrödinger equation, obtaining new exactly solvable multichannel potentials and, at the same time, opening the way for much wider applications of the J-matrix method. In particular, we will show that there exist exactly solvable potentials whose matrix representation has the form of a tridiagonal matrix whose non-zero entries are not numbers, but matrices.

The paper is organized as follows. In the following section, we show how an eigenvalue problem for a Hamiltonian possessing both external and internal degrees of freedom (i.e. a multichannel Hamiltonian) may be reduced to the eigenvalue problem for a difference operator having operator-valued coefficients. We introduce a special space of operator-valued sequences $\mathbb{T}\ell^2$, which is the counterpart of the Hilbert space ℓ^2. In this space we define an intertwining (or transformation) operator and its adjoint which relate solutions to two different operator-valued discrete eigenvalue problems. In section 3 we give a solution to the intertwining relation, specifying the transformation operator as well as the transformed Hamiltonian. We also show that in the particular case of a two-dimensional space for an internal degree of freedom, our formulae acquire the simplest, though nontrivial, form and we are able to establish a one-to-one correspondence between the spaces of solutions of the initial and transformed equations. Section 4 is devoted to applying our method to finding a wide class of new exactly solvable two-channel Hamiltonians whose matrix representation has infinite block-tridiagonal form. In the last section, we outline a possible continuation of this work.

2. Preliminaries

We will show here how the usual eigenvalue problem for a Hamiltonian with a composite interaction may be reduced to an eigenvalue problem for a difference operator having operator-valued (or matrix-valued) coefficients.

To begin with, let us consider a time-independent Hamiltonian $\mathcal{H}_0$ describing a composite system possessing both external and internal degrees of freedom. The external degrees of freedom are related to the motion of the system, considered as a material point, interacting with time-independent external fields. We shall deal only with stationary states and describe this motion with the help of a separable Hilbert space H which, in particular, may be chosen as the space $L^2(\mathbb{R})$. Internal degrees of freedom, related to the system's structure (spin, colour, charm, ...), will be associated with an N-dimensional linear space $\mathbb{C}^N$. A typical example is a multichannel quantum system [6]. Thus, the Hamiltonian $\mathcal{H}_0$ acts in the Hilbert space $\mathbb{H} = H \otimes \mathbb{C}^N$ and we consider the eigenvalue problem

$$\mathcal{H}_0|\Psi\rangle = \mathcal{E}|\Psi\rangle \qquad |\Psi\rangle \in \mathbb{H}. \tag{2.1}$$

Let $\{|n\rangle, n = 0, 1, 2, \ldots\}$ be an orthonormal basis in H and $\{e_\alpha, \alpha = 1, \ldots, N\}$ be an orthonormal basis in $\mathbb{C}^N$, so that the set $\{|n, \alpha\rangle \equiv |n\rangle e_\alpha = e_\alpha|n\rangle\}$ is a basis in the space $\mathbb{H}$. Hence, denoting the Fourier coefficients of an element $|\Psi\rangle \in \mathbb{H}$ by $\psi_{n,\alpha}$ in the basis $|n, \alpha\rangle$, one has

$$|\Psi\rangle = \sum_{n,\alpha} \psi_{n,\alpha}|n, \alpha\rangle = \sum_{n=0}^{\infty} \Psi_n|n\rangle \tag{2.2}$$

where $\Psi_n = \sum_{\alpha=1}^{N} \psi_{n,\alpha}e_\alpha$ denotes a column-vector of $\mathbb{C}^N$. Since the right-hand side of the equation (2.2) looks like a development of a vector $|\Psi\rangle$ over the basis $\{|n\rangle\}$, the quantities

$\{\Psi_n\}$ may be considered as the 'coordinates' of the ket $|\Psi\rangle$. Following this approach, we can consider the elements $|\Psi\rangle$ of $\mathbb{H}$, as a set of elements endowed with a different structure: it is not a vector space anymore, since its underlying space $\mathbb{C}^N$ is neither a field nor a ring (only the addition of this type of object is defined, but not a multiplication by the elements of $\mathbb{C}^N$); rather, it can be treated as a *module over the linear space* $\mathbb{C}^N$, which does not correspond to the usual algebraic construction of a module over a ring or a field [7]. We shall denote the set $\{|\Psi\rangle\}$ with this new structure by $\mathbb{T}$. Although the kets $\{|n\rangle\}$ are not elements of $\mathbb{T}$, we will call them a 'basis' in the sense that $\forall|\Psi\rangle \in \mathbb{T}$ there exists in $\mathbb{C}^N$ a set $\{\Psi_n\}$ such that (2.2) is valid, in the sense that it corresponds to a convergent Fourier series in $\mathbb{H}$. Note also that no notion of linear dependence or independence can be introduced in $\mathbb{T}$, since it is not a linear space, but a module. Nevertheless, one can introduce an inner product as a mapping $\mathbb{T} \times \mathbb{T} \to \mathbb{C}$ inherited from the inner product in $\mathbb{H}$, as follows:

$$\langle\Psi|\Phi\rangle := \sum_{n=0}^{\infty} \Psi_n^{\dagger}\Phi_n = \sum_{n=0}^{\infty}\sum_{\alpha=1}^{N} \psi_{n,\alpha}^{*}\varphi_{n,\alpha} \qquad \forall|\Psi\rangle, |\Phi\rangle \in \mathbb{T}. \tag{2.3}$$

Here, $\Psi_n^{\dagger}$ is the row-vector obtained by transposition and conjugation of the column-vector Ψ_n, and $\Psi_n^{\dagger}\Phi_n$ denotes the inner product of two elements of $\mathbb{C}^N$. By simplicity, we use the same notation $\langle\cdot|\cdot\rangle$ for the inner product in H and in $\mathbb{T}$. In terms of this inner product, one can define a distance between $|\Psi\rangle$ and $|\Phi\rangle$ as usual: $d(\Psi, \Phi) = [\langle\Psi - \Phi|\Psi - \Phi\rangle]^{1/2}$, which converts $\mathbb{T}$ to a metric space. As this metric generates a (strong) topology in $\mathbb{T}$, all the topological notions can be used for studying it, and it can be proved to be a complete metric space. From this point of view the right-hand side in (2.2) is a Fourier series convergent in the strong topology in $\mathbb{T}$. The fact that $\mathbb{T}$ is not a Hilbert space causes no difficulty from the physical point of view, since the underlying space $\mathbb{H}$ is a Hilbert space, and we introduce $\mathbb{T}$ only as a convenient tool.

In the following we shall impose the following form for the action of the operator $\mathcal{H}_0$ introduced in equation (2.1) on the basis $|n, \alpha\rangle$:

$$\mathcal{H}_0|n, \alpha\rangle = \sum_{\beta=1}^{N}[D_{n,\alpha,\beta}|n-1, \beta\rangle + D_{n+1,\alpha,\beta}|n+1, \beta\rangle + Q_{n,\alpha,\beta}|n, \beta\rangle]. \tag{2.4}$$

This type of action corresponds to a tridiagonal matrix whose non-zero entries are N-dimensional square matrices. The coefficients $D_{n,\alpha,\beta}$ and $Q_{n,\alpha,\beta}$ must satisfy $D_{n,\alpha,\beta} = D^{*}_{n,\beta,\alpha}$, $Q_{n,\alpha,\beta} = Q^{*}_{n,\beta,\alpha}$ to ensure the Hermitian character of $\mathcal{H}_0$.

If for a fixed value of n we consider the $N \times N$ matrices with entries $D_{n,\alpha,\beta}$ and $Q_{n,\alpha,\beta}$ as a matrix representation of the self-adjoint operators $\mathcal{D}_n = \mathcal{D}_n^{\dagger}$ and $\mathcal{Q}_n = \mathcal{Q}_n^{\dagger}$, whose action on $\mathbb{C}^N$ is

$$\mathcal{D}_n e_\alpha := \sum_{\beta=1}^{N} D_{n,\alpha,\beta} e_\beta \qquad \mathcal{Q}_n e_\alpha := \sum_{\beta=1}^{N} Q_{n,\alpha,\beta} e_\beta \tag{2.5}$$

then, according to (2.4), the action of the Hamiltonian $\mathcal{H}_0$ on the basis vector $|n, \alpha\rangle$ takes the form of a three-term relation with operator-valued coefficients

$$\mathcal{H}_0|n, \alpha\rangle = \mathcal{D}_n|n-1, \alpha\rangle + \mathcal{D}_{n+1}|n+1, \alpha\rangle + \mathcal{Q}_n|n, \alpha\rangle \qquad \mathcal{D}_0|n, \alpha\rangle = 0, \forall n, \forall\alpha.$$

We shall allow the operator $\mathcal{H}_0$ to be unbounded in $\mathbb{H}$, and to be defined on an appropriate dense set $\mathbb{D} \subset \mathbb{H}$, which is supposed to be invariant under the action of $\mathcal{H}_0$. Then, $\forall|\Psi\rangle \in \mathbb{D}$ the following Fourier series converges in $\mathbb{H}$

$$\mathcal{H}_0|\Psi\rangle = \sum_{n=0}^{\infty}\sum_{\alpha=1}^{N}[\psi_{n-1,\alpha}\mathcal{D}_n + \psi_{n+1,\alpha}\mathcal{D}_{n+1} + \psi_{n,\alpha}\mathcal{Q}_n]|n, \alpha\rangle. \tag{2.6}$$

With the help of this expression and (2.2) we can define the action of $\mathcal{H}_0$ in $\mathbb{T}$ to be

$$\mathcal{H}_0|\Psi\rangle = \sum_{n=0}^{\infty}(\mathcal{H}_0\Psi)_n|n\rangle \qquad (\mathcal{H}_0\Psi)_n := \mathcal{D}_{n+1}\Psi_{n+1} + \mathcal{D}_n\Psi_{n-1} + \mathcal{Q}_n\Psi_n. \tag{2.7}$$

Note that the right-hand side in the first equation is a convergent Fourier series in $\mathbb{T}$ since it just corresponds to the convergent Fourier series (2.6) in $\mathbb{H}$. Moreover, $\mathcal{H}_0$, as an operator acting in $\mathbb{T}$, has a well-defined domain of definition $\tilde{\mathbb{D}} \subset \mathbb{T}$ consisting of all elements of $\mathbb{D}$ considered as elements of $\mathbb{T}$. We note that since $\mathbb{D}$ is invariant under the action of $\mathcal{H}_0$, then $\tilde{\mathbb{D}}$ is also invariant under the action of $\mathcal{H}_0$ acting on $\mathbb{T}$.

The eigenvectors of $\mathcal{H}_0$ are obtained from the finite-difference eigenvalue problem

$$\mathcal{D}_{n+1}\Psi_{n+1} + \mathcal{D}_n\Psi_{n-1} + \mathcal{Q}_n\Psi_n = \mathcal{E}\Psi_n \tag{2.8}$$

with operator-valued coefficients (cf [8]). We do not indicate explicitly the dependence of Ψ_n on $\mathcal{E}$ in order to avoid cumbersome notation.

Let us suppose now that some solutions of the eigenvalue problem for $\mathcal{H}_0$ are known and we want to get a solution of the eigenvalue problem for another Hamiltonian $\mathcal{H}_1$. According to the usual strategy of intertwining operators [9], one can try to solve this problem by finding an intertwiner (or transformation operator) $\mathcal{L}$ defined by

$$\mathcal{L}\mathcal{H}_0 = \mathcal{H}_1\mathcal{L} \tag{2.9}$$

where we consider all operators acting on the same domain $\tilde{\mathbb{D}} \subset \mathbb{T}$. Once $\mathcal{L}$ is found, the eigenkets $|\tilde{\Psi}\rangle$ of $\mathcal{H}_1$ are obtained by applying $\mathcal{L}$ to the eigenkets $|\Psi\rangle$ of $\mathcal{H}_0$: $|\tilde{\Psi}\rangle = \mathcal{L}|\Psi\rangle$. Here, both $|\Psi\rangle$ and $|\tilde{\Psi}\rangle$ correspond to the same eigenvalue $\mathcal{E}$.

We consider now a particular ansatz for $\mathcal{L}$, by assuming that it acts on the basis element $|n,\alpha\rangle$ of the space $\mathbb{H}$ as follows:

$$\mathcal{L}|n,\alpha\rangle = \sum_{\beta=1}^{N}[A_{n,\alpha,\beta}|n-1,\beta\rangle + B_{n,\alpha,\beta}|n,\beta\rangle] \tag{2.10}$$

with $A_{0,\alpha,\beta} \equiv 0$. We easily find that the action of $\mathcal{L}$ on any $|\Psi\rangle$ from $\mathbb{D} \subset \mathbb{H}$ is

$$\mathcal{L}|\Psi\rangle = |\tilde{\Psi}\rangle = \sum_{n=0}^{\infty}\tilde{\Psi}_n|n\rangle \qquad \tilde{\Psi}_n \equiv (\mathcal{L}\Psi)_n := \mathcal{A}_{n+1}\Psi_{n+1} + \mathcal{B}_n\Psi_n. \tag{2.11}$$

Here $\mathcal{A}_n$ and $\mathcal{B}_n$ are the operators acting on $\mathbb{C}^N$, which in the basis $\{e_\alpha\}$ are represented by matrices with entries $A_{n,\alpha,\beta}$ and $B_{n,\alpha,\beta}$. For completeness, we remark that $\mathcal{A}_0 = 0$.

It should be mentioned that the second relations of (2.7) and (2.11) define the action of $\mathcal{H}_0$ and $\mathcal{L}$ on the space of vector-sequences $\{\Psi_n\}$, which we shall denote by $\mathbb{T}\ell^2$, since it is just a counterpart of the Hilbert space ℓ^2. To be precise, we should remark that these are not really the same operators as the ones defined above, since they act on different spaces. Nevertheless, we will use the same notation for them and for the inner product in $\mathbb{T}\ell^2$, believing that this will not cause any trouble to the reader. The inner product in $\mathbb{T}\ell^2$ is defined as usual by (2.3), where now $\Psi := \{\Psi_n\}$, $\Phi := \{\Phi_n\}$ are elements of $\mathbb{T}\ell^2$. Once we have an inner product in $\mathbb{T}\ell^2$, we are able to determine the adjoint $\mathcal{L}^\dagger$ of an operator $\mathcal{L}$, defined as usually by the relation $\langle\Psi|\mathcal{L}\Phi\rangle = \langle\mathcal{L}^\dagger\Psi|\Phi\rangle$, $\Psi, \Phi \in \mathbb{T}\ell^2$. We shall consider $\mathcal{L}^\dagger$ as acting on finite elements which form a dense set in $\mathbb{T}\ell^2$. Using equations (2.11), we evaluate $\langle\Psi|\mathcal{L}\Phi\rangle$ and find the action of $\mathcal{L}^\dagger$

$$(\mathcal{L}^\dagger\Psi)_n = \mathcal{A}_n^\dagger\Psi_{n-1} + \mathcal{B}_n^\dagger\Psi_n \tag{2.12}$$

where $\mathcal{A}_n^\dagger$ and $\mathcal{B}_n^\dagger$ are the operators adjoint to $\mathcal{A}_n$ and $\mathcal{B}_n$, respectively.

3. Discrete-matrix intertwining technique

3.1. Intertwining technique in the space $\mathbb{T}\ell^2$

The second relations of (2.7) and (2.11) define the action of $\mathcal{H}_0$ and $\mathcal{L}$ on any finite element $\{\Psi_n\}$ of $\mathbb{T}\ell^2$, meaning that the sequence $\{\Psi_n\}$ has a finite length. For the product $\mathcal{L}\mathcal{H}_0$ one gets

$$\mathcal{L}(\mathcal{H}_0\Psi)_n = \mathcal{A}_{n+1}(\mathcal{D}_{n+2}\Psi_{n+2} + \mathcal{Q}_{n+1}\Psi_{n+1} + \mathcal{D}_{n+1}\Psi_n) + \mathcal{B}_n(\mathcal{D}_{n+1}\Psi_{n+1} + \mathcal{Q}_n\Psi_n + \mathcal{D}_n\Psi_{n-1}). \tag{3.1}$$

A similar expression is obtained for $\mathcal{H}_1\mathcal{L}$. Then the intertwining relation (2.9) implies

$$\mathcal{A}_n\mathcal{D}_{n+1} = \tilde{\mathcal{D}}_n\mathcal{A}_{n+1} \tag{3.2}$$

$$\mathcal{B}_n\mathcal{D}_n = \tilde{\mathcal{D}}_n\mathcal{B}_{n-1} \tag{3.3}$$

$$\mathcal{A}_{n+1}\mathcal{Q}_{n+1} + \mathcal{B}_n\mathcal{D}_{n+1} = \tilde{\mathcal{D}}_{n+1}\mathcal{B}_{n+1} + \tilde{\mathcal{Q}}_n\mathcal{A}_{n+1} \tag{3.4}$$

$$\mathcal{A}_{n+1}\mathcal{D}_{n+1} + \mathcal{B}_n\mathcal{Q}_n = \tilde{\mathcal{D}}_n\mathcal{A}_n + \tilde{\mathcal{Q}}_n\mathcal{B}_n. \tag{3.5}$$

When solving these equations (the unknowns being the operators $\tilde{\mathcal{D}}_n$, $\tilde{\mathcal{Q}}_n$, $\mathcal{A}_n$, $\mathcal{B}_n$ and the data $\mathcal{D}_n$, $\mathcal{Q}_n$) we should take into account that $\mathcal{A}_0 = \mathcal{D}_0 = 0$. By assuming the operators $\mathcal{A}_n$, $\mathcal{B}_n$ to be invertible, we easily find from (3.3)

$$\tilde{\mathcal{D}}_n = \mathcal{B}_n\mathcal{D}_n(\mathcal{B}_{n-1})^{-1} \tag{3.6}$$

which being placed into (3.2) gives us

$$\mathcal{A}_n\mathcal{D}_{n+1}(\mathcal{A}_{n+1})^{-1} = \mathcal{B}_n\mathcal{D}_n(\mathcal{B}_{n-1})^{-1}. \tag{3.7}$$

Similarly, from (3.5) we have

$$\tilde{\mathcal{Q}}_n = \mathcal{A}_{n+1}\mathcal{D}_{n+1}(\mathcal{B}_n)^{-1} + \mathcal{B}_n\mathcal{Q}_n(\mathcal{B}_n)^{-1} - \tilde{\mathcal{D}}_n\mathcal{A}_n(\mathcal{B}_n)^{-1} \tag{3.8}$$

which together with (3.4) yields

$$\begin{aligned}\mathcal{A}_{n+1}\mathcal{Q}_{n+1} + \mathcal{B}_n\mathcal{D}_{n+1} = {}& \tilde{\mathcal{D}}_{n+1}\mathcal{B}_{n+1} + \mathcal{A}_{n+1}\mathcal{D}_{n+1}(\mathcal{B}_n)^{-1}\mathcal{A}_{n+1}\\ &+ \mathcal{B}_n\mathcal{Q}_n(\mathcal{B}_n)^{-1}\mathcal{A}_{n+1} - \tilde{\mathcal{D}}_n\mathcal{A}_n(\mathcal{B}_n)^{-1}\mathcal{A}_{n+1}.\end{aligned} \tag{3.9}$$

To solve this equation we proceed to eliminate the variable $\mathcal{B}_n$ by introducing a new auxiliary variable (actually, operator) σ_n defined by

$$\mathcal{B}_n = \mathcal{A}_{n+1}\sigma_n. \tag{3.10}$$

In equation (3.9) we replace $\tilde{\mathcal{D}}_n$ by its expression (3.6), multiply the result from the left by $(\mathcal{A}_{n+1})^{-1}$ and use (3.7) and (3.10) to rearrange the first term on the right-hand side to get a nonlinear finite-difference equation where the only unknown is σ_n:

$$\mathcal{Q}_{n+1} - \mathcal{D}_{n+2}\sigma_{n+1} - \mathcal{D}_{n+1}\sigma_n^{-1} = \sigma_n\big(\mathcal{Q}_n - \mathcal{D}_{n+1}\sigma_n - \mathcal{D}_n\sigma_{n-1}^{-1}\big)\sigma_n^{-1}. \tag{3.11}$$

This equation having operator-valued coefficients may be interpreted as a matrix finite-difference counterpart of the derivative of a Riccati equation. It can be linearized and 'integrated' by introducing the new variable $\mathcal{U}_n$ defined by

$$\sigma_n = -\mathcal{U}_{n+1}\mathcal{U}_n^{-1}. \tag{3.12}$$

Using it in (3.11) and simplifying, we get the difference equation $G_{n+1} = G_n$, with

$$G_n = \mathcal{U}_n^{-1}\big[\mathcal{Q}_n + \mathcal{D}_{n+1}\mathcal{U}_{n+1}\mathcal{U}_n^{-1} + \mathcal{D}_n\mathcal{U}_{n-1}\mathcal{U}_n^{-1}\big]\mathcal{U}_n \tag{3.13}$$

whose solution is $G_n = \Lambda$. The matrix Λ plays the role of an integration constant. This leads us to an equation for $\mathcal{U}_n$

$$\mathcal{D}_{n+1}\mathcal{U}_{n+1} + \mathcal{D}_n\mathcal{U}_{n-1} + \mathcal{Q}_n\mathcal{U}_n = \mathcal{U}_n\Lambda \tag{3.14}$$

which is identical to the eigenvalue problem (2.8), except for the fact that the vector-valued variable Ψ_n is replaced by the matrix-valued variable $\mathcal{U}_n$ and the scalar parameter $\mathcal{E}$ is replaced by the matrix Λ. We shall now show how solutions of (3.14) may be obtained from solutions of (2.8). Let us choose Λ to be diagonal: $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$. In fact, this is not a restriction, since this matrix is Hermitian (this follows from the Hermiticity of the matrices $\mathcal{D}$ and $\mathcal{Q}$) and therefore it can be diagonalized by an appropriate unitary transformation. Denote the columns of the matrix $\mathcal{U}_n$ by $U_{1,n}, \ldots, U_{N,n}$. If these vectors are solutions to the initial eigenvalue problem

$$\mathcal{D}_{n+1}U_{i,n+1} + \mathcal{D}_n U_{i,n-1} + \mathcal{Q}_n U_{i,n} = \lambda_i U_{i,n} \qquad i = 1, \ldots, N \tag{3.15}$$

then the matrix $\mathcal{U}_n$ is a solution to equation (3.14). Once the matrices $\mathcal{U}_n$ are given, one can evaluate $\mathcal{B}_n$ using (3.12) and (3.10), provided the matrices $\mathcal{A}_n$ are known.

Now, we shall obtain an expression for $\mathcal{A}_n$. In order to do that, we have to solve equation (3.7), where $\mathcal{B}_n$ are given in (3.10), that is

$$\mathcal{A}_n\mathcal{D}_{n+1}\mathcal{A}_{n+1}^{-1} = \mathcal{A}_{n+1}R_n\mathcal{A}_n^{-1}. \tag{3.16}$$

Note that the quantities

$$R_n = \sigma_n\mathcal{D}_n\sigma_{n-1}^{-1} \qquad n = 1, 2, \ldots \tag{3.17}$$

are known, because they depend only on the initial data $\mathcal{D}_n$ and the matrices σ_n (or $\mathcal{U}_n$), which are given. Using very simple algebra, one obtains from (3.7)

$$\mathcal{D}_{n+1}R_n = \left(\mathcal{A}_n^{-1}\mathcal{A}_{n+1}R_n\right)^2 \qquad R_n\mathcal{D}_{n+1} = \left(\mathcal{A}_{n+1}^{-1}\mathcal{A}_n\mathcal{D}_{n+1}\right)^2. \tag{3.18}$$

It is a simple exercise to see that these equations are compatible, which means that equation (3.16) has a nontrivial solution. Hence, using for instance the first one, we get

$$\mathcal{A}_{n+1} = \mathcal{A}_n(\mathcal{D}_{n+1}R_n)^{1/2}R_n^{-1}. \tag{3.19}$$

From this recursion relation we find all the matrices $\mathcal{A}_n$, $n = 2, 3, \ldots$, in terms of $\mathcal{A}_1$:

$$\mathcal{A}_n = \mathcal{A}_1\left[(\mathcal{D}_2R_1)^{1/2}R_1^{-1}\right]\left[(\mathcal{D}_3R_2)^{1/2}R_2^{-1}\right]\cdots\left[(\mathcal{D}_nR_{n-1})^{1/2}R_{n-1}^{-1}\right].$$

Once $\mathcal{A}_n$ and $\mathcal{B}_n$ are found, from (3.2) and (3.3) we have two equivalent expressions for $\tilde{\mathcal{D}}_n$:

$$\tilde{\mathcal{D}}_n = \mathcal{A}_{n+1}\sigma_n\mathcal{D}_n\sigma_{n-1}^{-1}\mathcal{A}_n^{-1} = \mathcal{A}_n\mathcal{D}_{n+1}\mathcal{A}_{n+1}^{-1}. \tag{3.20}$$

To find $\tilde{\mathcal{Q}}_n$ we use (3.5), which yields

$$\tilde{\mathcal{Q}}_n = \mathcal{A}_{n+1}\left(\mathcal{D}_{n+1} + \sigma_n\mathcal{Q}_n - \sigma_n\mathcal{D}_n\sigma_{n-1}^{-1}\right)\sigma_n^{-1}\mathcal{A}_{n+1}^{-1}. \tag{3.21}$$

It is necessary to mention that in finding the matrices $\tilde{\mathcal{D}}_n$ and $\tilde{\mathcal{Q}}_n$ we never required them to be self-adjoint. Therefore, our solution can provide us with both self-adjoint and non-self-adjoint matrices. The conditions $\tilde{\mathcal{D}}_n = \tilde{\mathcal{D}}_n^\dagger$ and $\tilde{\mathcal{Q}}_n = \tilde{\mathcal{Q}}_n^\dagger$ are really restrictions on the transformation function $\mathcal{U}_n$.

Note that the quantities $\{\mathcal{U}_n\}$ uniquely define both the transformation operator $\mathcal{L}$ and the new 'potentials' $\tilde{\mathcal{D}}_n$ and $\tilde{\mathcal{Q}}_n$. Therefore, we shall call $\mathcal{U} = \{\mathcal{U}_n\}$ the *transformation function*.

3.2. A particular case

In this section, we shall consider a particular nontrivial case corresponding to $N = 2$, where general formulae established in the previous section take the simplest form. Moreover, we shall show that for this case one can establish one-to-one correspondence between the spaces of solutions of the initial and transformed equations and investigate the factorization properties of transformation operators.

Let us put $N = 2$ and choose the matrices $\mathcal{D}_n$ and $\mathcal{Q}_n$ to be multiples of the 2×2 identity matrix $\mathcal{I}$: $\mathcal{D}_n = d_n\mathcal{I}$, $\mathcal{Q}_n = q_n\mathcal{I}$. Let u_{jn} be real solutions to the scalar difference equation

$$d_{n+1}u_{j,n+1} + d_n u_{j,n-1} + q_n u_{j,n} = \lambda_j u_{j,n} \qquad j = 1, 2. \tag{3.22}$$

Then, the matrix function

$$\mathcal{U}_n = \begin{pmatrix} u_{1,n} & -u_{2,n} \\ u_{1,n} & u_{2,n} \end{pmatrix} \tag{3.23}$$

is a solution to the initial eigenvalue problem (3.14), with $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2)$. We will show that it is suitable to be used as the transformation function in our algorithm, and that it will lead us to a nontrivial exactly solvable matrix-difference eigenvalue problem. After finding the inverse function

$$\mathcal{U}_n^{-1} = \Delta_n^{-1} \begin{pmatrix} u_{2,n} & u_{2,n} \\ -u_{1,n} & u_{1,n} \end{pmatrix} \tag{3.24}$$

where $\Delta_n = 2u_{1,n}u_{2,n}$, we get from (3.12)

$$\sigma_n = -\mathcal{U}_{n+1}\mathcal{U}_n^{-1} = -\Delta_n^{-1} \begin{pmatrix} \omega_n & \pi_n \\ \pi_n & \omega_n \end{pmatrix} \tag{3.25}$$

with

$$\omega_n = u_{1,n+1}u_{2,n} + u_{1,n}u_{2,n+1} \qquad \text{and} \qquad \pi_n = u_{1,n+1}u_{2,n} - u_{1,n}u_{2,n+1}. \tag{3.26}$$

Two remarkable properties of the matrices (3.25) should be mentioned:

(i) From equations (3.25) and (3.26), and from the real character of $u_{j,k}$, it is obvious that all the matrices σ_n are Hermitian: $\sigma_n^\dagger = \sigma_n$;

(ii) They form a commutative group with respect to matrix multiplication (this assertion can be trivially proved using (3.25)).

The last fact considerably simplifies our formulae. For instance, all the matrices in the expression for $\mathcal{A}_n$ commute, and we easily find

$$\mathcal{A}_n = \mathcal{D}_n^{1/2}\left(\mathcal{U}_{n-1}\mathcal{U}_n^{-1}\right)^{1/2} \tag{3.27}$$

where $\mathcal{A}_1$ has been chosen appropriately. The expression for $\mathcal{B}_n$ follows from (3.10):

$$\mathcal{B}_n = -\mathcal{D}_{n+1}^{1/2}\left(\mathcal{U}_{n+1}\mathcal{U}_n^{-1}\right)^{1/2}. \tag{3.28}$$

Using (3.20) and (3.21) we deduce formulae for $\tilde{\mathcal{D}}_n$ and $\tilde{\mathcal{Q}}_n$:

$$\begin{cases} \tilde{\mathcal{D}}_n = \left(\mathcal{D}_n\mathcal{D}_{n+1}\mathcal{U}_{n-1}\mathcal{U}_n^{-1}\mathcal{U}_{n+1}\mathcal{U}_n^{-1}\right)^{1/2} \\ \tilde{\mathcal{Q}}_n = \mathcal{Q}_n - \mathcal{D}_{n+1}\mathcal{U}_n\mathcal{U}_{n+1}^{-1} + \mathcal{D}_n\mathcal{U}_{n-1}\mathcal{U}_n^{-1}. \end{cases} \tag{3.29}$$

For our particular choice ($\mathcal{D}_n = d_n\mathcal{I}$, $\mathcal{Q}_n = q_n\mathcal{I}$), we get explicit expressions for these matrices in terms of the known solutions of (3.22). Indeed, using (3.23) and (3.24), and choosing the positive square root in the first of equations (3.29) we get $\tilde{\mathcal{D}}_n = \left(\begin{smallmatrix} a_+ & a_- \\ a_- & a_+ \end{smallmatrix}\right)$, where

$$a_\pm = \frac{\sqrt{d_n d_{n+1}}}{2}\left[\sqrt{\frac{u_{1,n+1}u_{1,n-1}}{u_{1,n}^2}} \pm \sqrt{\frac{u_{2,n+1}u_{2,n-1}}{u_{2,n}^2}}\right]. \tag{3.30}$$

Similarly, from the second of equations (3.29) one finds $\tilde{\mathcal{Q}}_n = \mathcal{Q}_n + \left(\begin{smallmatrix} b_+ & b_- \\ b_- & b_+ \end{smallmatrix}\right)$, where

$$b_\pm = \frac{d_n}{2}\left[\frac{u_{1,n-1}}{u_{1,n}} \pm \frac{u_{2,n-1}}{u_{2,n}}\right] - \frac{d_{n+1}}{2}\left[\frac{u_{1,n}}{u_{1,n+1}} \pm \frac{u_{2,n}}{u_{2,n+1}}\right]. \tag{3.31}$$

We note that for $u_{2,n} = u_{1,n}$ the matrices $\tilde{\mathcal{D}}_n$ and $\tilde{\mathcal{Q}}_n$ become diagonal, with the non-zero elements coinciding with those previously obtained for the scalar case [5].

From (3.30) and (3.31), it is clear that the transformed quantities are Hermitian provided $a_\pm$ and $b_\pm$ are real. On the other hand, it is easy to show that, under some reasonable assumptions on q_n and d_n, the operators $\mathcal{A}_n$ and $\mathcal{B}_n$ are anti-Hermitian. Indeed, from (3.14) we have

$$\mathcal{D}_{n+1}\mathcal{U}_{n+1}\mathcal{U}_n^{-1} + \mathcal{D}_n\mathcal{U}_{n-1}\mathcal{U}_n^{-1} = \mathcal{U}_n\Lambda\mathcal{U}_n^{-1} - \mathcal{Q}_n. \tag{3.32}$$

By choosing the matrix Λ appropriately, and provided the sequence $\{q_n\}$ is bounded from below, the matrices on both sides of (3.32) may be negative definite for all values of n to get the matrix $\mathcal{D}_{n+1}\sigma_n + \mathcal{D}_n\sigma_{n-1}$ positive definite. This is achieved by taking the operators $\mathcal{D}_n$, as well as σ_n, to be positive definite $\forall n$, and the anti-Hermiticity of $\mathcal{A}_n$ and $\mathcal{B}_n$ follows from (3.27) and (3.28).

Let us now calculate the superposition $\mathcal{L}^\dagger\mathcal{L}$ as an operator acting on $\mathbb{T}\ell^2$. Using equations (2.11), (2.12), (3.27) and (3.28), and the anti-Hermiticity of $\mathcal{A}_n$ and $\mathcal{B}_n$, we find

$$(\mathcal{L}^\dagger\mathcal{L}\Psi)_n = (\mathcal{D}_{n+1}\Psi_{n+1} + \mathcal{D}_n\Psi_{n-1} + \mathcal{Q}_n\Psi_n) - (\mathcal{D}_{n+1}\mathcal{U}_{n+1} + \mathcal{D}_n\mathcal{U}_{n-1} + \mathcal{Q}_n\mathcal{U}_n)\mathcal{U}_n^{-1}\Psi_n. \tag{3.33}$$

The first term on the right-hand side is simply the function $(\mathcal{H}_0\Psi)_n$ (see equation (2.7)). In the parentheses of the second term we see the action of the same operator on the transformation function $\mathcal{U}_n$. Hence, using (3.14) we transform the right-hand side of the previous equation to the form $(\mathcal{L}^\dagger\mathcal{L}\Psi)_n = (\mathcal{H}_0 - \mathcal{U}_n\Lambda\mathcal{U}_n^{-1})\Psi_n$. For the particular case of the transformation function given in (3.23) we easily obtain

$$\mathcal{U}_n\Lambda\mathcal{U}_n^{-1} = \frac{1}{2}\begin{pmatrix}\lambda_1+\lambda_2 & \lambda_1-\lambda_2\\ \lambda_1-\lambda_2 & \lambda_1+\lambda_2\end{pmatrix} =: \tilde{\Lambda} \tag{3.34}$$

and therefore

$$\mathcal{L}^\dagger\mathcal{L} = \mathcal{H}_0 - \tilde{\Lambda}. \tag{3.35}$$

We observe here a difference between our Darboux technique and supersymmetric quantum mechanics usually based on the factorization $\mathcal{L}^\dagger\mathcal{L} = \mathcal{H}_0 - \Lambda$. The factorization constant (in our case $\tilde{\Lambda}$) does not coincide with an eigenvalue of the transformation function. We believe that this is a manifestation of the fact that supersymmetric quantum mechanics based only on the factorization of the Hamiltonian is a particular case of the Darboux transformation method, which in general allows us to factorize other symmetry operators. A similar factorization takes place for the inverse superposition $\mathcal{L}\mathcal{L}^\dagger = \mathcal{H}_1 - \tilde{\Lambda}$ which may be established by applying the operator $\mathcal{L}$ to formula (3.35).

Our results indicate the existence of a one-to-one correspondence between the solution spaces of the initial and transformed equations. To develop this, we find the second solution $\hat{\mathcal{U}}_n$ of (3.14) with a given value of Λ, $\mathcal{D}_{n+1}\hat{\mathcal{U}}_{n+1} + \mathcal{D}_n\hat{\mathcal{U}}_{n-1} + \mathcal{Q}_n\hat{\mathcal{U}}_n = \hat{\mathcal{U}}_n\Lambda$. By eliminating $\mathcal{Q}_n$ from this equation and using the adjoint form of (3.14), we get $\mathcal{U}_n^\dagger\mathcal{D}_{n+1}\hat{\mathcal{U}}_{n+1} - \mathcal{U}_{n+1}^\dagger\mathcal{D}_{n+1}\hat{\mathcal{U}}_n + \mathcal{U}_n^\dagger\mathcal{D}_n\hat{\mathcal{U}}_{n-1} - \mathcal{U}_{n-1}^\dagger\mathcal{D}_n\hat{\mathcal{U}}_n = \mathcal{U}_n^\dagger\hat{\mathcal{U}}_n\Lambda - \Lambda\mathcal{U}_n^\dagger\hat{\mathcal{U}}_n$. Because of our particular choice of the transformation function, the right-hand side of the last equation vanishes and the resulting difference equation can be 'integrated' to yield

$$\mathcal{U}_{n-1}^\dagger\mathcal{D}_n\hat{\mathcal{U}}_n - \mathcal{U}_n^\dagger\mathcal{D}_n\hat{\mathcal{U}}_{n-1} = \mathcal{U}_n^\dagger\mathcal{D}_{n+1}\hat{\mathcal{U}}_{n+1} - \mathcal{U}_{n+1}^\dagger\mathcal{D}_{n+1}\hat{\mathcal{U}}_n = W_0.$$

From here we see that the matrix W_0 plays the role of the Wronskian for our discrete eigenvalue problem, and we find the recursion relation for $\hat{\mathcal{U}}_n$, which being iterated gives

$$\hat{\mathcal{U}}_n = \left(\mathcal{U}_0^\dagger\right)^{-1}\mathcal{U}_n^\dagger\hat{\mathcal{U}}_0 + \sum_{k=1}^{n}\mathcal{D}_k^{-1}\left(\mathcal{U}_k^\dagger\right)^{-1}\mathcal{U}_n^\dagger\left(\mathcal{U}_{k-1}^\dagger\right)^{-1}W_0. \tag{3.36}$$

Now, we can act with $\mathcal{L}$ on this function in order to get a solution of the transformed equation with eigenvalue Λ. After some algebraic manipulation we arrive at

$$\mathcal{S}_n \equiv (\mathcal{L}\hat{\mathcal{U}})_n = \mathcal{D}_{n+1}^{-1/2}\big(\mathcal{U}_n\mathcal{U}_{n+1}^{-1}\big)^{1/2}\big(\mathcal{U}_n^{\dagger}\big)^{-1} \tag{3.37}$$

where the constant matrix W_0 has been chosen to be the identity. We note also that the operator-valued function $\mathcal{S}_n$ belongs to the kernel of the operator $\mathcal{L}^{\dagger}$, i.e. $\mathcal{L}^{\dagger}\mathcal{S}_n = 0$. This can be established easily by using formulae (2.12), (3.27) and (3.28). Once $\mathcal{S}_n$ is found, one can get the second solution $\hat{\mathcal{S}}_n$ of the transformed equation by using (3.36) with appropriate changes. The solution $\hat{\mathcal{U}}_n$ also corresponds to the matrix eigenvalue Λ.

The operator $\mathcal{L}^{\dagger}$ satisfies the intertwining relation adjoint to (2.9). This means that it is a transformation operator from solutions of the transformed equation to solutions of the initial one. Then, if $E \neq \lambda_{1,2}$, the function $\tilde{\Psi}_n = (\mathcal{L}\Psi)_n$ is a nontrivial solution of the transformed equation, whereas $\Phi_n = (\mathcal{L}^{\dagger}\tilde{\Psi})_n$ is also a nontrivial solution, but to the initial equation. Solutions of the transformed equation with eigenvalues $\lambda_{1,2}$ can be found using (3.36) and (3.37). Hence, we have constructed a one-to-one correspondence between the solution spaces of the initial and transformed eigenvalue problems.

4. Application

We shall use the above developed discrete-matrix Darboux transformation to generate new exactly solvable two-channel potentials. Indeed, we will show that even starting with the simplest case of the two-channel uncoupled free particle Hamiltonian, we can obtain a family of nontrivial coupled exactly solvable potentials. In contradistinction to all known exactly solvable potentials, they are represented by infinite block-tridiagonal matrices. The eigenvalue problem for such matrices is not solvable by the usual methods.

Consider the free particle Hamiltonian $h_0 = p_x^2$. Since the momentum operator p_x is expressible in terms of the harmonic oscillator ladder operators $a^+ = \mathrm{i}\,\mathrm{d}/\mathrm{d}x + \mathrm{i}x/2$ and $a = \mathrm{i}\,\mathrm{d}/\mathrm{d}x - \mathrm{i}x/2$, $h_0 = (a+a^+)^2/4$ is a quadratic form in a and a^+. Therefore, the action of h_0 on the oscillator basis $|n\rangle$, which in the coordinate representation is

$$\psi_n(x) = \langle x|n\rangle = (-\mathrm{i})^n\big(n!2^n\sqrt{2\pi}\big)^{-1/2}\,\mathrm{e}^{-x^2/4}H_n\big(x/\sqrt{2}\big) \tag{4.1}$$

where $H_n(z)$ are Hermite polynomials, takes the form of a three-term relation

$$h_0|n\rangle = d_n|n-2\rangle + d_{n+2}|n+2\rangle + q_n|n\rangle \qquad d_n = \frac{\sqrt{n(n-1)}}{4} \quad q_n = \frac{n}{2} + \frac{1}{4}. \tag{4.2}$$

To derive (4.2) we have used $a|n\rangle = \sqrt{n}|n-1\rangle$ and $a^+|n\rangle = \sqrt{n+1}|n+1\rangle$. Let $|\psi_E\rangle$ be a continuous spectrum eigenket of h_0: $h_0|\psi_E\rangle = E|\psi_E\rangle$. Then, using the Hermiticity of h_0 we get the scalar discrete eigenvalue problem

$$d_n\psi_{n-2} + d_{n+2}\psi_{n+2} + q_n\psi_n = E\psi_n. \tag{4.3}$$

A 'physical' solution $\psi_n = \psi_n(E) = \langle\psi_E|n\rangle$, $E > 0$, to this equation can be easily obtained since it coincides with the Fourier image of function (4.1):

$$\psi_n(E) = 2\big(n!2^n\sqrt{2\pi}\big)^{-1/2}\,\mathrm{e}^{-E}H_n\big(\sqrt{2E}\big). \tag{4.4}$$

Consider now an uncoupled two-channel problem with the diagonal matrix Hamiltonian $\mathcal{H}_0 = h_0\mathcal{I}$. This is only the kinetic energy operator which acts in the Hilbert space $\mathbb{H} = H \otimes \mathbb{C}^2$, with basis $|n,\alpha\rangle = e_\alpha|n\rangle$, $e_1 = (1,0)^t$, $e_2 = (0,1)^t$. The action of $\mathcal{H}_0$ on a two-component vector $|\Psi\rangle = \sum_{n=0}^{\infty}\Psi_n|n\rangle$, is given by (2.7), where

$$(\mathcal{H}_0\Psi)_n = \mathcal{D}_{n+2}\Psi_{n+2} + \mathcal{D}_n\Psi_{n-2} + \mathcal{Q}_n\Psi_n \in \mathbb{C}^2 \tag{4.5}$$

$\mathcal{D}_n = d_n\mathcal{I}$ and $\mathcal{Q}_n = q_n\mathcal{I}$. Since the solutions of the one-channel problem for h_0 (4.3) are known, we know solutions of the two-channel problem for $\mathcal{H}_0$. In particular, two-column vectors $\left\{\Psi_n^+(E)\right\} = \{(\psi_n(E), 0)^t\}$ and $\{\Psi_n^-(E)\} = \{(0, \psi_n(E))^t\}$ with $\psi_n(E)$ given in (4.4) are solutions to equation

$$\mathcal{D}_{n+2}\Psi_{n+2} + \mathcal{D}_n\Psi_{n-2} + \mathcal{Q}_n\Psi_n = \mathcal{E}\Psi_n \tag{4.6}$$

with $\mathcal{E} = E$. The matrix $\mathcal{U}_n$ given in (3.23), with matrix elements

$$u_{1n} = (n!2^n)^{-1/2}H_n\big(\sqrt{2\lambda_1}\big) \qquad u_{2n} = (n!2^n)^{-1/2}H_n\big(\sqrt{2\lambda_2}\big) \tag{4.7}$$

$\lambda_{1,2} < 0$, is a solution to the matrix eigenvalue problem

$$\mathcal{D}_{n+2}\mathcal{U}_{n+2} + \mathcal{D}_n\mathcal{U}_{n-2} + \mathcal{Q}_n\mathcal{U}_n = \mathcal{U}_n\Lambda \tag{4.8}$$

with $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2)$. Remark that this two-channel problem is not exactly of the type considered in section 2: here the eigenvalue problem involves shifting by two units the discrete variable n. Nevertheless, it is not difficult to see that we can consider here two independent problems, one for the even and the other for the odd values of n. For the sake of simplicity we omit these details and give below only the final results.

Now let the Hamiltonian $\mathcal{H}_1 = \mathcal{H}_0 + \mathcal{V}$ have the interaction potential

$$\mathcal{V}|n,\alpha\rangle = \mathcal{G}_n|n-2,\alpha\rangle + \mathcal{G}_{n+2}|n+2,\alpha\rangle + \mathcal{R}_n|n,\alpha\rangle \qquad \alpha = 1, 2 \tag{4.9}$$

with coefficients $\mathcal{G}_n$ and $\mathcal{R}_n$ that are to be determined. Let $|\tilde{\Psi}\rangle$ be an eigenvector of $\mathcal{H}_1$, $\mathcal{H}_1|\tilde{\Psi}\rangle = \mathcal{E}|\tilde{\Psi}\rangle$, and $\{\tilde{\Psi}_n\}$ be its Fourier coefficients over the basis $|n\rangle$. Let these quantities satisfy the similar discrete-matrix eigenvalue problem

$$\tilde{\mathcal{D}}_n\tilde{\Psi}_{n-2} + \tilde{\mathcal{D}}_{n+2}\tilde{\Psi}_{n+2} + \tilde{\mathcal{Q}}_n\tilde{\Psi}_n = \mathcal{E}\tilde{\Psi}_n. \tag{4.10}$$

Suppose also that the unspecified quantities $\mathcal{G}_n$ and $\mathcal{R}_n$ are related to $\tilde{\mathcal{D}}_n$ and $\tilde{\mathcal{Q}}_n$ by

$$\tilde{\mathcal{D}}_n = \mathcal{G}_n + \frac{1}{4}\sqrt{n(n-1)}\mathcal{I} \qquad \tilde{\mathcal{Q}}_n = \mathcal{R}_n + \left(\frac{n}{2} + \frac{1}{4}\right)\mathcal{I}. \tag{4.11}$$

Consider the subclass of potentials (4.9) for which equation (4.10) coincides with the Darboux transform of the initial eigenvalue problem (4.6) and express the transformation function $\mathcal{U}_n$ in the form (3.23) with the entries given in (4.7). All these assumptions are satisfied if the functions $\tilde{\mathcal{D}}_n$ and $\tilde{\mathcal{Q}}_n$ have the form

$$\begin{cases} \tilde{\mathcal{D}}_n = \left[\mathcal{D}_n\mathcal{D}_{n+2}\mathcal{U}_{n-2}\mathcal{U}_n^{-1}\mathcal{U}_{n+2}\mathcal{U}_n^{-1}\right]^{1/2} \\ \tilde{\mathcal{Q}}_n = \mathcal{Q}_n - \mathcal{D}_{n+2}\mathcal{U}_n\mathcal{U}_{n+2}^{-1} + \mathcal{D}_n\mathcal{U}_{n-2}\mathcal{U}_n^{-1}. \end{cases} \tag{4.12}$$

The matrices $\mathcal{G}_n$ and $\mathcal{R}_n$ represent in this case the 'potential differences' $\mathcal{G}_n = \tilde{\mathcal{D}}_n - \mathcal{D}_n$, $\mathcal{R}_n = \tilde{\mathcal{Q}}_n - \mathcal{Q}_n$ which are expressed in terms of known quantities only. Solutions of (4.10) are found with the aid of (2.11), $\mathcal{A}_n$ and $\mathcal{B}_n$ given in (3.27) and (3.28):

$$\tilde{\Psi}_n = \mathcal{D}_{n+2}^{1/2}\big(\mathcal{U}_n\mathcal{U}_{n+2}^{-1}\big)^{1/2}\big(\Psi_{n+2} - \mathcal{U}_{n+2}\mathcal{U}_n^{-1}\Psi_n\big). \tag{4.13}$$

Thus, we have solved the eigenvalue problem for a Hamiltonian $\mathcal{H}_1 = \mathcal{H}_0 + \mathcal{V}$ having in the basis $|n,\alpha\rangle$ an interaction in the form of the block-tridiagonal matrix

$$\langle k\beta|\mathcal{V}|n\alpha\rangle = G_{\beta,\alpha,n}\delta_{n-2,k} + G_{\beta,\alpha,n+2}\delta_{n+2,k} + R_{\beta,\alpha,n}\delta_{n,k}. \tag{4.14}$$

$G_{\beta,\alpha,n}$ and $R_{\beta,\alpha,n}$ are entries for $\mathcal{G}_n$ and $\mathcal{R}_n$ respectively, and are found from (4.11) and (4.12). Hence, using (3.30) and the notation $H_n(\sqrt{2\lambda_1}) = H_{n,1}$ and $H_n(\sqrt{2\lambda_2}) = H_{n,2}$, we find

$$\mathcal{G}_n = -\frac{1}{4}\sqrt{n(n-1)}\mathcal{I} + \begin{pmatrix} a_+ & a_- \\ a_- & a_+ \end{pmatrix} \qquad \text{and} \qquad \mathcal{R}_n = \begin{pmatrix} b_+ & b_- \\ b_- & b_+ \end{pmatrix}$$

with

$$
\begin{aligned}
a_\pm &= \frac{\sqrt{n(n-1)}}{8}\left[\sqrt{\frac{H_{n+2,1}H_{n-2,1}}{(H_{n,1})^2}} \pm \sqrt{\frac{H_{n+2,2}H_{n-2,2}}{(H_{n,2})^2}}\right] \\
b_\pm &= \frac{n(n-1)}{4}\left[\frac{H_{n-2,1}}{H_{n,1}} \pm \frac{H_{n-2,2}}{H_{n,2}}\right] - \frac{(n+1)(n+2)}{4}\left[\frac{H_{n,1}}{H_{n+2,1}} \pm \frac{H_{n,2}}{H_{n+2,2}}\right].
\end{aligned}
\tag{4.15}
$$

Note that the quantities $H_{n,1}$ and $H_{n,2}$ are real for n even and purely imaginary for n odd and therefore $\mathcal{G}_n$ and $\mathcal{R}_n$ are real in either case.

Let us find now the asymptotic behaviour of these quantities with respect to the discrete variable n. For this purpose we need an appropriate asymptotic expression for the Hermite polynomials, which can be found with the help of the relationship between Hermite and Laguerre polynomials, $H_{2n}(z) = (-1)^n 2^{2n} n! L_n^{-1/2}(z^2)$, and using the series expansion for the Laguerre polynomials (see [10], equations 10.15(4), 10.15(5)) in which we keep only the two first terms and consider $z^2 < 0$, $\mathrm{Im}(z) > 0$. We then have

$$
L_n^{-1/2}(z^2) = \frac{(2n-1)!!}{2^{n+1}n!}\exp(z^2/2 - 2\mathrm{i}z\sqrt{n})\left[1 - \frac{z^2}{16n} + O(n^{-3/2})\right] \tag{4.16}
$$

and

$$
H_{2n}(\sqrt{2\lambda}) = \frac{(-2)^n(2n-1)!!}{2}\exp(\lambda - 2\mathrm{i}\sqrt{2\lambda n})\left[1 - \frac{\lambda}{8n} + O(n^{-3/2})\right]. \tag{4.17}
$$

From (4.15) we obtain $a_+ = n/4 + 1/8 - 1/(32n) + O(1/n^2)$, $b_+ = 1/2 + O(1/n^2)$, $a_- = O(1/n^2)$ and $b_- = O(1/n^2)$. The matrices $\mathcal{G}_n$ and $\mathcal{R}_n$ become diagonal up to terms of order $1/n^2$. Moreover, up to quadratic terms in $1/n$, a_+ coincides with d_{n+1} and $(q_n + b_+)$ with q_{n+1}. This means that for $n \to \infty$ the Darboux transformed potentials $\tilde{\mathcal{D}}_n$ and $\tilde{\mathcal{Q}}_n$ coincide with the initial ones, up to the shifting $n \to n+1$. This is the discrete multichannel analogue of a property of the usual Darboux transformation saying that usually the potential difference vanishes asymptotically.

Now we will estimate the asymptotics of solutions of the transformed discrete eigenvalue problem. To do it we will assemble the column-vectors $\Psi_n^+(E_1)$ and $\Psi_n^-(E_2)$ in the matrix Ξ_n, which is a matrix solution of the initial equation corresponding to the diagonal matrix eigenvalue with entries E_1 and E_2. The matrix $\tilde{\Xi}_n := \big(\tilde{\Psi}_n^+(E_1), \tilde{\Psi}_n^-(E_2)\big)$ (the columns $\tilde{\Psi}_n^\pm(E_{1,2})$ are given by (4.13) with the replacement of Ψ_n by $\Psi_n^\pm(E_{1,2})$, respectively) is the corresponding matrix solution to the transformed equation

$$
\tilde{\Xi}_n = \mathcal{P}\Xi_n \qquad \mathcal{P} = \mathcal{D}_{n+2}^{1/2}\big(\mathcal{U}_{n+2}\mathcal{U}_n^{-1}\big)^{1/2}\big(\mathcal{U}_n\mathcal{U}_{n+2}^{-1}\Xi_{n+2}\Xi_n^{-1} - \mathcal{I}\big). \tag{4.18}
$$

Then, using the same asymptotics for the Hermite polynomials (4.17), we obtain

$$
\begin{aligned}
&\mathcal{P} = \begin{pmatrix} p_1 & q \\ q & p_2 \end{pmatrix} \\
&p_{1,2} = \sqrt{E_{1,2}} - \tfrac{1}{2}\big(\sqrt{\lambda_1} + \sqrt{\lambda_2}\big) + o(n^{-1/2}) \qquad q = \tfrac{1}{2}\big(\sqrt{\lambda_2} - \sqrt{\lambda_1}\big) + o(n^{-1/2}).
\end{aligned}
\tag{4.19}
$$

We see that for $n \to \infty$ the matrix $\mathcal{P}$ does not depend on the variable n, a result which agrees with the fact that the potential difference vanishes asymptotically. This means, in particular, that if we choose the initial waves propagating only in one direction in both channels, the same asymptotic behaviour will be exhibited by the solutions of the transformed equation. This property is precisely a two-channel analogue of the transparency for the transformed potential we have obtained.

5. Conclusion

In this paper, we have introduced a difference intertwining operator for discrete Schrödinger equations with operator-valued coefficients and then we studied some of its basic properties, such as the factorization and the possibility of establishing a one-to-one correspondence between the spaces of solutions of the initial and transformed equations. This approach allowed us to get new exactly solvable eigenvalue problems for block-tridiagonal matrices. By working out the particular example of a two-channel Schrödinger equation, we have found a wide class of exactly solvable two-channel potentials represented by block-tridiagonal matrices.

One of the possible applications of the method we have developed may be in describing the scattering of composite particles, such as nucleons or atoms, in the frame of the multichannel scattering theory [6]. This possibility is supported by the known applications of the J-matrix method for describing the one-channel scattering [3, 4]. Precisely in this line, the next step of this work will be to find the transformation of the scattering data for the spectral problem on a semi-axis.

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